LIMITING WEAK TYPE ESTIMATE FOR CAPACITARY MAXIMAL FUNCTION

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Abstract. A capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy-Littlewood maximal function of an $L^1(\mathbb{R}^n)$-function (cf. [3, 4]) is discovered.

1. Statement of Theorem

For an L^1_{loc}-integrable function f on $\mathbb{R}^n$, $n \geq 1$, let $Mf(x)$ denote the Hardy-Littlewood maximal function of f at $x \in \mathbb{R}^n$:

$$Mf(x) = \sup_{x\in B} \frac{1}{\mathscr{L}(B)} \int_B |f(y)|dy,$$

where the supremum is taken over all Euclidean balls B containing x and $\mathscr{L}(B)$ stands for the n-dimensional Lebesgue measure of B. Among several results of [3, 4], P. Janakiraman obtained the following fundamental limit:

$$\lim_{\lambda\to 0} \lambda \mathscr{L}(\{x \in \mathbb{R}^n : \ Mf(x) > \lambda\}) = \|f\|_1 = \int_{\mathbb{R}^n} |f(y)|dy \quad \forall \quad f \in L^1(\mathbb{R}^n).$$

This note studies the limiting weak type estimate for a capacity. To be more precise, recall that a set function $C(\cdot)$ on $\mathbb{R}^n$ is said to be a capacity (cf. [1, 2]) provided that

$$\begin{cases} C(\emptyset) = 0; \\ 0 \leq C(A) \leq \infty \quad \forall \quad A \subseteq \mathbb{R}^n; \\ C(A) \leq C(B) \quad \forall \quad A \subseteq B \subseteq \mathbb{R}^n; \\ C(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty C(A_i) \quad \forall \quad A_i \subseteq \mathbb{R}^n. \end{cases}$$

For a given capacity $C(\cdot)$ let

$$M_C f(x) = \sup_{x\in B} \frac{1}{C(B)} \int_B |f(y)|dy$$

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be the capacitary maximal function of an L^1_{loc}-integrable function f at x for which the supremum ranges over all Euclidean balls B containing x; see also [5].

In order to establish a capacitary analogue of the last limit formula for $f \in L^1(\mathbb{R}^n)$, we are required to make the following natural assumptions:

- Assumption 1 - the capacity $C(B(x,r))$ of the ball $B(x,r)$ centered at x with radius r is a function depending on r only, but also the capacity $C(\{x\})$ of the set $\{x\}$ of a single point $x \in \mathbb{R}^n$ equals 0.
- Assumption 2 - there are two nonnegative functions ϕ and ψ on $(0,\infty)$ such that

$$\begin{cases}\phi(t)C(E) \le C(tE) \le \psi(t)C(E) \ \ \forall \ \ t>0 \ \ \& \ \ tE = \{tx \in \mathbb{R}^n : \ x \in E \subseteq \mathbb{R}^n\};\\ \lim_{t\to 0}\phi(t) = 0 = \lim_{t\to 0}\psi(t) \ \ \& \ \ \lim_{t\to 0}\psi(t)/\phi(t) = \tau \in (0,\infty).\end{cases}$$

Here, it is worth mentioning that the so-called p-capacity satisfies all the assumptions; see also [6].

Theorem 1.1. *Under the above-mentioned two assumptions, one has*

$$\lim_{\lambda\to 0}\lambda C(\{x \in \mathbb{R}^n : M_C f(x) > \lambda\}) \approx \|f\|_1 \quad \forall \quad f \in L^1(\mathbb{R}^n).$$

Here and henceforth, $\mathsf{X} \approx \mathsf{Y}$ *means that there is a constant* $c > 0$ *independent of* X *and* Y *such that* $c^{-1}\mathsf{Y} \le \mathsf{X} \le c\mathsf{Y}$.

2. Four Lemmas

To prove Theorem 1.1, we will always suppose that $C(\cdot)$ is a capacity obeying Assumptions 1-2 above, but also need four lemmas based on the following capacitary maximal function $M_C\nu$ of a finite nonnegative Borel measure ν on $\mathbb{R}^n$:

$$M_C\nu(x) = \sup_{B\ni x}\frac{\nu(B)}{C(B)} \quad \forall \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$ containing x.

Lemma 2.1. *If δ_0 is the delta measure at the origin, then*

$$C(\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\}) = \frac{1}{\lambda}.$$

Proof. According to the defintion of the delta measure and Assumptions 1-2, we have

$$M_C\delta_0(x) = \frac{1}{C(B(x,|x|))} \quad \forall \quad |x| \neq 0.$$

Now, if x obeys $M_C\delta_0(x) > \lambda$, then

$$C(B(x,|x|)) < \frac{1}{\lambda}.$$

Note that if $C(B(0,r))$ equals $\frac{1}{\lambda}$, then one has the following property:

$$\begin{cases} C(B(x,|x|)) < \frac{1}{\lambda} & \forall \quad |x| < r; \\ C(B(x,|x|)) = \frac{1}{\lambda} & \forall \quad |x| = r; \\ C(B(x,|x|)) > \frac{1}{\lambda} & \forall \quad |x| > r. \end{cases}$$

Therefore,

$$\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\} = B(0,r),$$

and consequently,

$$C(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda\}) = C(B(0,r)) = \frac{1}{\lambda}.$$

□

Lemma 2.2. *If ν is a finte nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$, then*

$$\lim_{t\to 0} C(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\}) = \frac{1}{\lambda},$$

where

$$\begin{cases} t > 0; \\ \nu_t(E) = \nu(\frac{1}{t}E); \\ \frac{1}{t}E = \{\frac{x}{t} : \ x \in E\}; \\ E \subseteq \mathbb{R}^n. \end{cases}$$

Proof. For two positive numbers ϵ and η, choose ϵ_1 small relative to both ϵ and η, but also let t be small and the induced ϵ_t be such that

$$\begin{cases} \nu_t(B(0,\epsilon_t)) > 1-\epsilon; \\ \epsilon_t = 3^{-1}\epsilon_1; \\ \lim_{t\to 0}\epsilon_t = 0; \\ \epsilon < \eta C(B(0,\epsilon_1)). \end{cases}$$

Now, if

$$\begin{cases} E^t_{1,\lambda} = \Big\{x \in \mathbb{R}^n \setminus B(0,\epsilon_1) : \ \lambda < M_C\nu_t(x) \le \frac{1}{C\left(B(x,|x|-\epsilon_t)\right)}\Big\}; \\ E^t_{2,\lambda} = \Big\{x \in \mathbb{R}^n \setminus B(0,\epsilon_1) : \ \max\{\lambda, \frac{1}{C\left(B(x,|x|-\epsilon_t)\right)}\} < M_C\nu_t(x)\Big\}, \end{cases}$$

then

$$E^t_{1,\lambda} \cup E^t_{2,\lambda} \cup B(0,\epsilon_1) = \{x \in \mathbb{R}^n : M_C\nu_t(x) > \lambda\}.$$

On the one hand, for such $x \in E^t_{2,\lambda}$ and $\forall \tilde{r} > 0$ that

$$\frac{\nu_t(B(x,\tilde{r}))}{C(B(x,|x|-\epsilon_t))} \le \frac{1}{C(B(x,|x|-\epsilon_t))} < M_C\nu_t(x).$$

Additionally, since for any $r_1,\ r_2$ satisfying $0 \le r_1 \le r_2$,

$$C(B(x, r_1)) \le C(B(x, r_2)),$$

$C(B(x, r))$ is an increasing function with respect to r. There exists $r < |x| - \epsilon_t$ such that

$$\frac{\nu_t(B(x, r))}{C(B(x, |x| - \epsilon_t))} \le \frac{\nu_t(B(x, r))}{C(B(x, r))} \le M_C \nu_t(x),$$

and hence by the Assumption 1, for any $x_i \in E^t_{2,\lambda}$ there exists $r_i > 0$ such that

$$r_i < |x_i| - \epsilon_t \ \ \& \ \ \lambda \le \frac{\nu_t(B(x_i, r_i))}{C(B(x, r))}.$$

By the Wiener covering lemma, there exists a disjoint collection of such balls $B_i = B(x_i, r_i)$ and a constant $\alpha > 0$ such that

$$\cup_i B_i \subseteq E^t_{2,\lambda} \subseteq \cup_i \alpha B_i,$$

Therefore, we get a constant $\gamma > 0$, which only depends on α, such that

$$C(E^t_{2,\lambda}) \le \gamma \sum_i C(B_i) < \gamma \sum_i \frac{\nu_t(B_i)}{\lambda} \le \frac{\gamma \epsilon}{\lambda},$$

thanks to

$$B_i \cap B(0, \epsilon_t) = \emptyset \ \ \& \ \ 1 - \nu_t(B(0, \epsilon_t)) < \epsilon.$$

On the other hand, if $x \in E^t_{1,\lambda}$, then

$$\begin{aligned} \frac{1 - \epsilon}{C(B(x, |x| + \epsilon_t))} &\le \frac{\nu_t(B(x, |x| + \epsilon_t))}{C(B(x, |x| + \epsilon_t))} \\ &\le M_C \nu_t(x) \\ &\le \frac{1}{C(B(x, |x| - \epsilon_t))}. \end{aligned}$$

Since

$$\begin{cases} \lim_{t \to 0} \left(\frac{1}{C\left(B(x, |x| + \epsilon_t)\right)} - \frac{1}{C\left(B(x, |x| - \epsilon_t)\right)} \right) = 0; \\ \lim_{t \to 0} \left(\frac{1}{C\left(B(x, |x| + \epsilon_t)\right)} - \frac{1}{C\left(B(x, |x|)\right)} \right) = 0, \end{cases}$$

for $\eta > 0$ there exists $T > 0$ such that

$$\begin{aligned} |M_C \nu_t(t) - M_C \delta_0| &< \eta + \frac{\epsilon}{C(B(0, |x|))} \\ &< \eta + \frac{\epsilon}{C(B(0, \epsilon_1))} \\ &< 2\eta \quad \forall\ t \in (0, T). \end{aligned}$$

Note that

$$M_C \delta_0(x) - 2\eta \le M_C \nu_t \le M_C \delta_0(x) + 2\eta \quad \forall \quad x \in E^t_{1,\lambda}.$$

Thus

$$\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\} \subseteq E^t_{1,\lambda} \subseteq \{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}.$$

This in turn implies

$$\begin{aligned} & C(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}) \\ \leq \ & C(E^t_{1,\lambda}) \\ \leq \ & C(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}). \end{aligned}$$

Now, an application of Lemma 2.1 yields

$$\frac{1}{\lambda + 2\eta} \leq C\Big(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\} \cap (\mathbb{R}^n \setminus B(0, \epsilon_1))\Big) \leq \frac{1}{\lambda - 2\eta} + \frac{\gamma\epsilon}{\lambda}.$$

Letting $t \to 0$ and using Assumption 1, we get

$$\lim_{t\to 0} C(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\}) = \frac{1}{\lambda}.$$

□

Lemma 2.3. *If ν is a nonnegative Borel measure on $\mathbb{R}^n$, then $M_C\nu(x)$ is upper semi-continuous.*

Proof. According to the definition of $M_C\nu(x)$, there exists a radius r corresponding to $M_C\nu(x) > \lambda > 0$ such that

$$\frac{\nu(B(x, r))}{C(B(x, r))} > \lambda.$$

For a slightly larger number s with $\lambda + \delta > s > r$, we have

$$\frac{\nu(B(x, r))}{C(B(x, s))} > \lambda.$$

Then applying Assumption 1, for any z satisfying $|z - x| < \delta$,

$$M_C\nu(z) \geq \frac{\nu(B(z, s))}{C(B(z, s))} \geq \frac{\nu(B(x, r))}{C(B(x, s))} > \lambda.$$

Thereby, the set $\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}$ is open, as desired. □

Lemma 2.4. *If ν is a finite nonnegative Borel measure on $\mathbb{R}^n$, then there exists a constant $\gamma > 0$ such that*

$$\lambda C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > \lambda\}) \leq \gamma\nu(\mathbb{R}^n).$$

Proof. Following the argument for [7, Page 39, Theorem 5.6], we set $E_\lambda = \{x \in \mathbb{R}^n : \ M_C\nu(x) > \lambda\}$, and then select a ν-measurable set $E \subseteq E_\lambda$ with $\nu(E) < \infty$. Lemma 2.3 proves that E_λ is open. Therefore, for each $x \in E$, there exists an x-related ball B_x such that

$$\frac{\nu(B_x)}{C(B_x)} > \lambda.$$

A slight modification of the proof of [7, Page 39, Lemma 5.7] applied to the collection of balls $\{B_x\}_{x\in E}$, and Assumption 2, show that we can find a sub-collection of disjoint balls $\{B_i\}$ and a constant $\gamma > 0$ such that

$$C(E) \le \gamma \sum_i C(B_i) \le \sum_i \frac{\gamma}{\lambda}\nu(B_i) \le \frac{\gamma}{\lambda}\nu(\mathbb{R}^n).$$

Note that E is an arbitrary subset of E_λ. Thereby, we can take the supremum over all such E and then get

$$C(E_\lambda) < \frac{\gamma}{\lambda}\nu(\mathbb{R}^n).$$

□

3. Proof of Theorem

First of all, suppose that ν is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$. According to the definition of the capacitary maximal function, we have

$$M_C\nu_t(x) = \sup_{r>0}\frac{\nu_t(B(x,r))}{C(B(x,r))} = \sup_{r>0}\frac{\nu(B(\frac{x}{t},\frac{r}{t}))}{C(tB(\frac{x}{t},\frac{r}{t}))}.$$

From Assumption 2 it follows that

$$\frac{M_C\nu(\frac{x}{t})}{\psi(t)} \le M_C\nu_t(x) \le \frac{M_C\nu(\frac{x}{t})}{\phi(t)},$$

and such that

$$\begin{aligned}\Big\{x\in\mathbb{R}^n:\ M_C\nu(\frac{x}{t}) > \lambda\psi(t)\Big\} \ &\subseteq\ \Big\{x\in\mathbb{R}^n:\ M_C\nu_t(x) > \lambda\Big\}\\ &\subseteq\ \Big\{x\in\mathbb{R}^n:\ M_C\nu(\frac{x}{t}) > \lambda\phi(t)\Big\}.\end{aligned}$$

The last inclusions give that

$$\begin{aligned}&\frac{\phi(t)}{\psi(t)}\lambda\psi(t)C(\{x\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\psi(t)\})\\ &\quad\le \lambda\phi(t)C(\{x\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\psi(t)\})\\ &\quad\le \lambda C(\{tx\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\psi(t)\})\\ &\quad= \lambda C(\{x\in\mathbb{R}^n:\ M_C\nu(x/t) > \lambda\psi(t)\})\\ &\quad\le \lambda C(\{x\in\mathbb{R}^n:\ M_C\nu_t(x) > \lambda\})\\ &\quad\le \lambda C(\{x\in\mathbb{R}^n:\ M_C\nu(x/t) > \lambda\phi(t)\})\\ &\quad= \lambda C(\{tx\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\phi(t)\})\\ &\quad\le \lambda\psi(t)C(\{x\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\phi(t)\})\\ &\quad\le \frac{\psi(t)}{\phi(t)}\lambda\phi(t)C(\{x\in\mathbb{R}^n:\ M_C\nu(x) > \lambda\phi(t)\}).\end{aligned}$$

These estimates and Lemma 2.2, plus applying Assumption 2 and letting $t \to 0$, in turns derive

$$\begin{aligned}\tau^{-1} &\leq \liminf_{\lambda\to 0} \lambda C(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}) \\ &\leq \limsup_{\lambda\to 0} \lambda C(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}) \leq \tau.\end{aligned}$$

Next, let

$$h(\lambda) = \lambda C(\{x \in \mathbb{R}^n : M_C\nu > \lambda\}).$$

By Lemma 2.4 and the last estimate for both the limit inferior and the limit superior, there exists two constants $A > 0$ and $\lambda_0 > 0$ such that

$$A \leq h(\lambda) \leq \gamma \quad \forall \quad \lambda \in (0, \lambda_0).$$

Moreover, for any given $\varepsilon > 0$, choose a sequence $\{y_i = [\frac{\gamma}{A}(1-\varepsilon)^N]^i\}_1^\infty$, where N is a natural number satisfying $\frac{\gamma}{A}(1-\varepsilon)^N < 1$. Then, there exists an integer $N_0 \geq 1$, such that $y_{N_0} < \lambda_0$. Hence, for any $n > m > N_0$ we have

$$\begin{aligned}&|h(y_m) - h(y_n)| \\ &\leq |y_m C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > y_m\}) - y_n C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > y_n\})| \\ &\leq |y_m - y_n| C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > y_m\}) \\ &\quad + y_n |C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > y_m\}) - C(\{x \in \mathbb{R}^n : \ M_C\nu(x) > y_n\})| \\ &\leq |y_m - y_n| \frac{\gamma}{y_m} + y_n \left|\frac{\gamma}{y_n} - \frac{A}{y_m}\right| \\ &\leq \gamma(1 - \frac{y_n}{y_m}) + (\gamma - A\frac{y_n}{y_m}) \\ &\leq \gamma(1 - [\frac{\gamma}{A}(1-\varepsilon)^N]^{n-m}) + (\gamma - A[\frac{\gamma}{A}(1-\varepsilon)^N]^{n-m}) \\ &\leq \gamma(1 - (1-\varepsilon)^{N(n-m)}) + (\gamma - \gamma(1-\varepsilon)^{N(n-m)}) \\ &\leq 2\gamma N(n-m)\varepsilon.\end{aligned}$$

Consequently, $\{h(y_i)\}$ is a Cauchy sequence, $D = \lim_{i\to\infty} h(y_i)$ exists. Note that for any small λ there exists a large i such that

$$y_{i+1} \leq \lambda \leq y_i.$$

Thereby, from the triangle inequality it follows that if i is large enough then

$$\begin{aligned} |h(\lambda) - D| &\leq |h(\lambda) - h(y_i)| + |h(y_i) - D| \\ &\leq |y_i - \lambda|\frac{\gamma}{y_i} + \lambda|\frac{\gamma}{\lambda} - \frac{A}{y_i}| + |h(y_i) - D| \\ &\leq \gamma(1 - \frac{\lambda}{y_i}) + (\gamma - A\frac{\lambda}{y_i}) + |h(y_i) - D| \\ &\leq \gamma(1 - \frac{y_{i+1}}{y_i}) + (\gamma - A\frac{y_{i+1}}{y_i}) + |h(y_i) - D| \\ &\leq (2\gamma N + 1)\varepsilon \end{aligned}$$

This in turn implies that $\lim_{\lambda\to 0}\lambda C(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\})$ exists, and consequently,

$$\tau^{-1} \leq \lim_{\lambda\to 0}\lambda C(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}) \leq \tau$$

holds.

Finally, upon employing the given $L^1(\mathbb{R}^n)$ function f with $\|f\|_1 > 0$ to produce a finite nonnegative measure ν with $\nu(\mathbb{R}^n) = 1$ via

$$\nu(E) = \frac{1}{\|f\|_1}\int_E |f(y)|dy \quad \forall \quad E \subseteq \mathbb{R}^n,$$

we obtain

$$\lim_{\lambda\to 0}\lambda C(\{x \in \mathbb{R}^n : \ M_C f(x) > \lambda\|f\|_1\}) \approx 1,$$

thereby getting

$$\lim_{\lambda\to 0}\lambda\|f\|_1 C(\{x \in \mathbb{R}^n : \ M_C f(x) > \lambda\|f\|_1\}) \approx \|f\|_1.$$

By setting $\tilde{\lambda} = \lambda\|f\|_1$ in the last estimate, we reach the desired result.

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